

FIR Filter Design: Hilbert Transformer's Example

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Abstract

A Finite Impulse Response (FIR) filter can be created by applying simple shifting and truncation on the analog prototype filter. Although shifting is a simple operation, proper care must be taken while applying the discrete-time shifting property, which ceases to follow for a fractional shift. Here we take the example of a Hilbert Transformer to confirm this fact. The MATLAB scripts are included in the appendix.

1 Introduction

The ideal frequency response of a digital Hilbert transformer is:

$$H(\Omega) = \begin{cases} -j & \Omega > 0 \\ 0 & \Omega = 0 \\ j & \Omega < 0 \end{cases} = \begin{cases} -j \operatorname{sgn}(\Omega) & \Omega \neq 0 \\ 0 & \Omega = 0 \end{cases} \quad (1)$$

The relation used to compute the inverse DTFT of $H(\Omega)$ is given as:

$$h_n = \frac{1}{2\pi} \int_{2\pi} H(\Omega) e^{j\Omega n} d\Omega \quad (2)$$

Plugging $H(\Omega)$ from Equation (1) into (2):

$$h_n = \frac{1}{2\pi} \left(\int_{-\pi}^0 j e^{j\Omega n} d\Omega + \int_0^{\pi} -j e^{j\Omega n} d\Omega \right) = \begin{cases} \frac{2}{\pi n} & n - \text{odd} \\ 0 & n - \text{even} \end{cases}$$

Shifting and Truncation are necessary steps for converting this non-causal filter to a realizable FIR filter. In short,

- Shifting helps capture most of the power in the frequency spectrum
- Truncation enables one to implement this filter on a digital piece of hardware viz. a microcontroller or a DSP

Following the DTFT shifting property,

$$h[n - N_s] \Leftrightarrow H(\Omega) e^{-j\Omega N_s} \quad \forall \quad N \in Z$$

As specified in the lab handout, the ideal response is shifted by $N_s = (N - 1)/2$. Here is the catch, $N_s \notin Z$ for an even value of N . That will be a fractional shift, and DTFT shifting property doesn't apply in this case. Hence, we can only apply the shifting property for odd values of N .

Two Kinds of Odd Integers

We define two types of odd integers which N can take thus affecting $(N - 1)/2$. Consider the following,

Type I: For numbers like 1, 5, 9, ..., $(N - 1)/2$ is even

Type II: For numbers like 3, 7, 11, ..., $(N - 1)/2$ is odd

Equations (3) and (4) summarize $h_{\text{shift}, \text{odd}-N}$ for type-I and type-II respectively.

$$h_{\text{shift}, T1-\text{odd}-N} = \begin{cases} \frac{2}{\pi \{n - (N-1)/2\}} & n - \text{odd} \\ 0 & n - \text{even} \end{cases} \quad (3)$$

$$h_{\text{shift}, T2-\text{odd}-N} = \begin{cases} \frac{2}{\pi \{n - (N-1)/2\}} & n - \text{even} \\ 0 & n - \text{odd} \end{cases} \quad (4)$$

The next section derives $H_{\text{shift}, N}$ in a more robust fashion, one which works for any value of N (even or odd).

2 A More General Approach

Plugging $H_{shift,N}[\Omega]$ from Lab#7 handout into Equation (2),

$$\begin{aligned} \Rightarrow h_{shift,N} &= \frac{1}{2\pi} \left[\int_{-\pi}^0 j e^{-j\Omega(N-1)/2} e^{jn\Omega} d\Omega + \int_0^{\pi} -j e^{-j\Omega(N-1)/2} e^{jn\Omega} d\Omega \right] \\ \Rightarrow \frac{j}{2\pi} &\left[\int_{-\pi}^0 e^{j\Omega(n-(N-1)/2)} d\Omega - \int_0^{\pi} e^{-j\Omega(n-(N-1)/2)} d\Omega \right] \\ \Rightarrow \left(\frac{j}{2\pi(n-(N-1)/2)} \right) &\left[e^{j\Omega(n-(N-1)/2)} \Big|_{-\pi}^0 - e^{j\Omega(n-(N-1)/2)} \Big|_0^{\pi} \right] \\ \Rightarrow \left(\frac{1}{2\pi(n-(N-1)/2)} \right) &\left[1 - e^{-j\pi(n-(N-1)/2)} - e^{j\pi(n-(N-1)/2)} + 1 \right] \end{aligned}$$

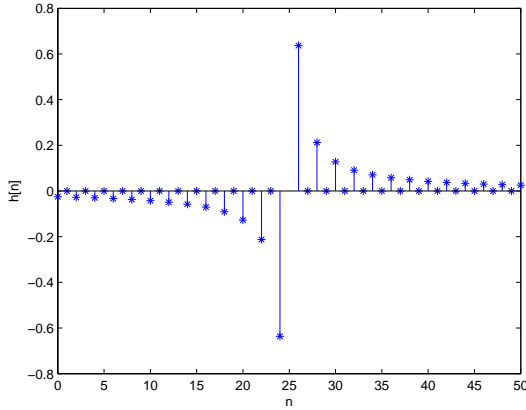
Equation (5) is the most general equation, which applies to both odd and even values of N.

$$h_{shift,N} = \left(\frac{1}{\pi(n-(N-1)/2)} \right) \left(1 - \cos \left(\left(n - \frac{N-1}{2} \right) \pi \right) \right) \quad (5)$$

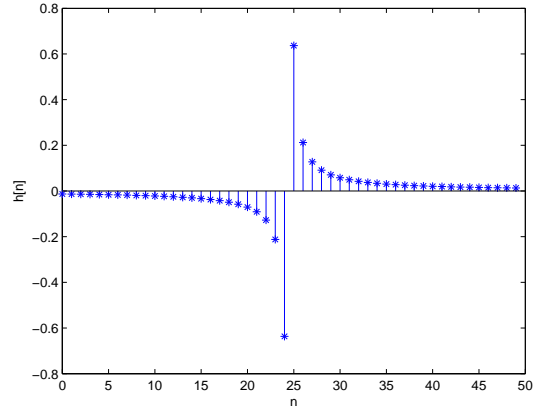
Looking at the previous result in light of N being odd/even:

Care must be taken while reading 'n' and 'N' because we are trying to analyze both of them in odd/even perspective. For N as a odd integer we reconfirm the expression in Equations (3) and (4).

However, for N being an even integer, there is no such simplification possible (Use Equation (4)) and as it turns out, every other element is not zero as it is in the odd-N case. An odd length Hilbert Transformer will therefore be computationally more efficient than an even length one. The following figure depicts the case for length 51 and 50.



(a) Length-51 Hilbert Transformer



(b) Length-50 Hilbert Transformer

Figure 1: FIR Filter Design: The Hilbert Transformer

3 Conclusion

Hence we discovered the associated caveats of shifting and truncating the ideal response in order to achieve the desired FIR response. In this case, the truncation was done with a simple rectangular or “boxcar” ($\Pi(\cdot)$) window. The analyses with other windows take similar forms. With shifting, we found the possibilities of fractional sample shifts, and ways to get around with a general method.

A Computing the Filter Coefficients

Listing 1: MATLAB Script for Computing Hilbert FIR Filter-Coefficients

```
N = 50;
n = 0:N-1;
h = (1 ./ (pi*(n-(N-1)/2))) .* (1-cos(pi*(n-(N-1)/2)));
figure, stem(n, h, 'k'), xlabel('n'), ylabel('h[n]')
```

B Writing a Header File

Listing 2: MATLAB Script for Writing a Header File

```
% Assuming h is in the workspace
fid = fopen('header.h','wt');
fprintf(fid, 'float h[]={ ');
fprintf(fid, '%12g,',h); %12g specifies the precision
fprintf(fid, '};');
fprintf(fid, '\n');
fclose(fid);
```